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SIMULTANEOUS DEVELOPMENT OF THE LAMINAR VELOCITY AND TEMPERATURE FIELDS IN A CIRCULAR DUCT FOR THE TEMPERATURE BOUNDARY CONDITION OF THE THIRD KIND

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Abstract—To analyze the influence of the developing flow in a circular duct on the laminar forced convection heat transfer, the non-linear momentum and linear energy equation are solved successively by employing the Galerkin–Kantorowich method of variational calculus. Assuming constant fluid properties, negligible axial diffusion and temperature boundary condition of the third kind, a closed form solution for velocity and a semi analytic solution for temperature are derived. It is concluded that there can be a considerable difference, depending upon Biot number and Prandtl number, between the local Nusselt number considering the radial convection and that neglecting it.

NOMENCLATURE

A, channel cross section;
Bi, Biot number, equation (5);
$\mathbf{F}$, a vector, equation (11);
Nu, Nusselt number, equation (35);
Pe, Peclet number, equation (5);
Pr, Prandtl number, equation (5);
R, tube radius;
Re, Reynolds number, equation (5);
T, temperature;
c_p, specific heat at constant pressure;
d_h, hydraulic diameter;
h, heat-transfer coefficient, equation (33);
k, overall heat-transfer coefficient, equation (4);
m, a parameter, equation (10);
p, pressure;
q, heat flux;
r, radial coordinate;
t, time;
v, velocity;
x, axial coordinate.

Greek symbols

λ, thermal conductivity;
ν, kinematic viscosity;
ρ, mass density;
$[\ ]$, matrix;
$\{\ \}$, column vector.

Subscripts

a, ambient;
d, fully developed;
ent, entrance region, equation (26);
i, running index;
inf, asymptotic value;
j, running index;
m, mean value;
r, radial coordinate direction;
w, wall;
x, axial coordinate direction;
0, value for $x = 0$.

Superscripts

$\bar{\ }$, dimensionless quantity, equation (5);
$'$, $(\mathrm{d}/\mathrm{d}\bar{x})$.

1. INTRODUCTION

In the most of the analyses on the laminar forced convection heat transfer in a channel, either the boundary condition of the first kind characterized by the prescribed wall temperature or the boundary condition of the second kind expressed by the prescribed wall heat flux is assumed. A more realistic condition in many applications, however, will be the temperature boundary condition of the third kind: the local wall heat flux is a linear function of the local wall temperature. This situation is relatively less studied and is encountered in the heat transfer process, where the radiative heat transfer, describable in terms of Newton's law of cooling, occurs at the channel wall.

The main concern of the present analysis is with the laminar forced convection heat transfer in the thermal entrance region of a circular duct for the temperature boundary condition of the third kind. Accordingly, some typical studies on this particular subject are mentioned below.

Fully developed viscous flow

Schenk and Dumore [1] solved the Sturm–Liouville type eigenvalue problem and determined the first three eigenvalues for the different wall resistance parameters. Sideman *et al.* [2] extended the analysis [1] by evaluating the first five eigenvalues. Using the finite difference method, Mckillop *et al.* [3] analyzed the same problem for Newtonian and non-Newtonian fluids. Hsu [4] refined the analysis [2] by considering the heat conduction of the fluid in the direction of the flow and by evaluating the first ten eigenvalues.

Slug flow

Considering the axial fluid heat conduction, Schneider [5] solved the problem formulated in [1] and determined the first six eigenvalues for a flat channel and a circular duct. Golos [6] obtained an exact series solution by neglecting the axial heat conduction and by employing the Laplace transformation. He also derived an approximate solution by applying the principles of restricted variation. Tyagi and Nigam [7] solved the same problem by utilizing the Galerkin method of variational calculus.

The assumption of the axially constant velocity profile introduced in the analyses mentioned is removed in this paper by considering the simultaneous development of the velocity and temperature fields.

The objective of the present paper is to investigate the laminar forced convection heat transfer in a circular duct for the temperature boundary condition of the third kind. Three different types of the flow situation are treated: slug flow, fully developed viscous flow and developing flow. To solve the non-linear momentum equation, the Galerkin–Kantorowich method of variational calculus is employed. The non-linearity of the momentum equation is kept in its original form and is not violated. A complete closed form solution is obtained for the developing velocity field. The influence of the radial convection on the local Nusselt number for various Prandtl and Biot numbers is investigated. Since the axial molecular momentum transport and the heat conduction are neglected, this analysis is valid for $Re \geqslant 50$ and $Pe \geqslant 50$ only.

2. ANALYSIS

Consider steady, developing laminar flow in a circular duct. Assuming constant fluid properties and neglecting the heat generation within the fluid, the laminar forced heat convection subject to boundary condition of the third kind for temperature can be described by the equations

$$\operatorname{div} \mathbf{v} = 0, \tag{1}$$

$$\rho(Dv_x/Dt) = -\operatorname{grad} p + \rho\nu \operatorname{div} \operatorname{grad} v_x, \tag{2}$$

$$\rho c_p(DT/Dt) = \lambda \operatorname{div} \operatorname{grad} T, \tag{3}$$

$$\left.\begin{aligned} &x = 0: v_x = v_{x,0},\ v_r = 0,\ p = p_0,\ T = T_0,\\ &x \to \infty: T \to T_a,\ v_x \to v_{x,d},\ v_r \to 0,\\ &r = 0: \partial T/\partial r = 0,\ \partial v_x/\partial r = 0,\\ &r = R: v_x = 0,\ v_r = 0,\ \lambda(\partial T/\partial r) + k(T - T_a) = 0, \end{aligned}\right\} \tag{4}$$

where k is the overall-heat-transfer coefficient based on the resistance of wall and ambient side surface resistance and T_a is the constant ambient temperature. For an easy treatment of the governing equations, the following dimensionless quantities are introduced:

$$\left.\begin{aligned} &\bar{x} = x/R,\ \bar{r} = r/R,\\ &\bar{v}_x = v_x/v_{x,m},\ \bar{v}_r = v_r/v_{x,m},\ v_{x,m} = (1/A)\int_A v_x \, dA,\\ &\bar{p} = (p - p_0)/(\rho v_{x,m}^2),\ \bar{T} = (T - T_a)/(T_0 - T_a),\\ &Re = v_{x,m} R/\nu,\ Pr = \nu\rho c_p/\lambda,\\ &Pe = RePr = v_{x,m} R\rho c_p/\lambda,\ Bi = Rk/\lambda. \end{aligned}\right\} \tag{5}$$

Neglecting the molecular momentum transport and the heat conduction in the direction of flow, from the equations given above, one can obtain

$$(\partial\bar{v}_x/\partial\bar{x}) + \bar{v}_r/\bar{r} + (\partial\bar{v}_r/\partial\bar{r}) = 0, \tag{6}$$

$$L(\bar{v}_x) \equiv Re[\bar{v}_x(\partial\bar{v}_x/\partial\bar{x}) + \bar{v}_r(\partial\bar{v}_x/\partial\bar{r}) + (d\bar{p}/d\bar{x})] - [(1/\bar{r})(\partial\bar{v}_x/\partial\bar{r}) + (\partial^2\bar{v}_x/\partial\bar{r}^2)] = 0, \tag{7}$$

$$L(\bar{T}) \equiv Pe[\bar{v}_x(\partial\bar{T}/\partial\bar{x}) + \bar{v}_r(\partial\bar{T}/\partial\bar{r})] - [(1/\bar{r})(\partial\bar{T}/\partial\bar{r}) + (\partial^2\bar{T}/\partial\bar{r}^2)] = 0. \tag{8}$$

The conditions describing the problem are:

$$\left.\begin{aligned} &\bar{x} = 0: \bar{v}_x = 1,\ \bar{v}_r = 0,\ \bar{p} = 0,\ \bar{T} = 1,\\ &\bar{x} \to \infty: \bar{T} \to 0,\ \bar{v}_r \to 0,\ \bar{v}_x \to \bar{v}_{x,d} = 2(1 - \bar{r}^2),\\ &\bar{r} = 0: (\partial\bar{T}/\partial\bar{r}) = 0,\ (\partial\bar{v}_x/\partial\bar{r}) = 0,\\ &\bar{r} = 1: \bar{v}_x = \bar{v}_r = 0,\ (\partial\bar{T}/\partial\bar{r}) + Bi\bar{T} = 0. \end{aligned}\right\} \tag{9}$$

To solve equations (7) and (8), the Galerkin–Kantorowich method of variational calculus is employed, which allows to reduce a partial differential equation to an ordinary one. The method is well described by Kantorowitsch and Krylow [8] and Krajewski [9]. Let the approximate velocity and temperature field be

$$\bar{v}_x = (1 + 2/m)(1 - |\bar{r}|^m), \quad m = m(\bar{x}), \tag{10a}$$

$$\bar{v}_r = (dm/d\bar{x})(\bar{r}/m^2)(1 - |\bar{r}|^m + m|\bar{r}|^m \ln|\bar{r}|), \tag{10b}$$

$$\bar{T} = \sum_j f_j(\bar{x})\{1 - C_j[(j+1)^2\bar{r}^{2j} - j^2\bar{r}^{2j+2}]\}, \tag{11a}$$

$$C_j = Bi/[Bi(2j+1) + 2j(j+1)], \tag{11b}$$

$$\{\mathbf{F}\} = \{f_1(\bar{x}), f_2(\bar{x}), \ldots f_N(\bar{x})\}. \tag{11c}$$

The fields (10) and (11) satisfy the continuity equation (6), the symmetry and boundary conditions (9) and energy equation (8) at the duct wall. Taking the momentum equation (7) and the energy equation (8) with the natural boundary conditions (9) as the Euler equation of the variational formulation, one may solve

$$\int_A L(\bar{v}_x)\,\delta\bar{v}_x \, dA = 0, \tag{12}$$

$$\int_A L(\bar{T})\,\delta\bar{T} \, dA = 0, \tag{13}$$

to evaluate the unknown functions $m(\bar{x})$ and $f_j(\bar{x})$. To determine the value of the unknown functions at the channel entrance, one may employ the conditions

$$\int_A g_1(\bar{r})[\bar{v}_x(\bar{x} = 0) - \bar{v}_{x,0}]^2 \, dA \to \min, \tag{14}$$

$$\int_A g_2(\bar{r})[\bar{T}(\bar{x} = 0) - \bar{T}_0]^2 \, dA \to \min, \tag{15}$$

where g_1 and g_2 are appropriate weighting functions. For the present problem

$$g_1 = 1 \quad \text{and} \quad g_2 = \bar{v}_x(\bar{x} = 0) \tag{16}$$

are suggested.

The important feature of the present method is that the non-linear momentum equation is solved in its original form without simplifying it. This is in contrast to the analysis performed by Langhaar [10] and Sparrow *et al.* [11].

Since the fluid properties are constant, the momentum and energy equations can be solved successively. With

$$\delta\bar{v}_x = (\partial\bar{v}_x/\partial m)\delta m = -(1/m^2)[2(1-|\bar{r}|^m) + m(m+2)|\bar{r}|^m \ln|\bar{r}|]\,\delta m, \quad (17)$$

$$\int_0^1 \delta\bar{v}_x \bar{r}\,d\bar{r} = 0, \quad (18)$$

from equations (7), (10), (12) and (14), one can derive

$$\bar{x} \to 0 : m = m_0 \to \infty, \quad (19)$$

$$\bar{x} \to \infty : m = m_{inf} \to 2, \quad (dm/d\bar{x}) \to 0, \quad (20)$$

$$Re(dm/d\bar{x}) = -(m+2)(m-2)(m+1)^3(3m+2)^3/[m^2(12m^3+97m^2+144m+60)], \quad (21)$$

$$(\bar{x}/Re) = -(13/216)\ln(1-2/m) + \ln(1+2/m) - (49/27)\ln(1+1/m) + (7/8)\ln[1+2/(3m)] - (23/9)/(m+1) + (1/6)/(m+1)^2 + 5/(3m+2) - 2/(3m+2)^2. \quad (22)$$

To calculate $p(\bar{x})$, one may use either

$$\int_0^1 L(\bar{v}_x)\bar{r}\,d\bar{r} = 0, \quad (23)$$

or

$$\int_0^1 L(\bar{v}_x)\bar{v}_x\bar{r}\,d\bar{r} = 0. \quad (24)$$

From equations (7), (10), (21) and (24), one obtains

$$-Re(d\bar{p}/d\bar{x}) = (1+2/m)^2 \times \left[m + \frac{3(7m+6)(m-2)(m+1)(3m+2)}{2(12m^3+97m^2+144m+60)}\right]. \quad (25)$$

Defining

$$\bar{p} = \bar{p}_{ent} + Re(d\bar{p}/d\bar{x})_{inf}(\bar{x}/Re), \quad (26)$$

it follows

$$(d\bar{p}_{ent}/dm) = [Re(d\bar{p}/d\bar{x}) - Re(d\bar{p}/d\bar{x})_{inf}]/[Re(dm/d\bar{x})]. \quad (27)$$

From equations (21), (25) and (27), one can derive

$$(d\bar{p}_{ent}/dm) = (87m^5+557m^4+920m^3+744m^2 + 432m+144)/[2(m+2)(m+1)^3(3m+2)^3], \quad (28)$$

$$\bar{p}_{ent} = 81n(1+2/m) - 101n(1+1/m) + 21n[1+2/(3m)] - (41/2)/(m+1) + (3/2)/(m+1)^2 + (104/3)/(3m+2) - (64/3)/(3m+2)^2. \quad (29)$$

In this way, the laminar entrance flow is described fully analytically. With the analytical expression (10) for the velocity field, the energy equation can be solved easily. The discussion of the entrance flow results is given in [12]. There, the approximation (10) was successfully used to analyze the magneto-hydrodynamic channel flow heat transfer in the thermal entrance region.

To determine the temperature field $\bar{T}(\bar{x}, \bar{r})$, the approximations (10) and (11) are inserted in equation (8). With

$$\delta\bar{T} = \sum_j (\partial\bar{T}/\partial f_j)\delta f_j, \quad (30)$$

from equations (8) and (13), one can derive a system of ordinary differential equations for $f_j(\bar{x})$ as follows (for details see Appendix)

$$RePr[D]\{\mathbf{F}'\} - RePr(m'/m^2)[V]\{\mathbf{F}\} = [W]\{\mathbf{F}\}. \quad (31)$$

Combining equations (8), (11) and (15), one can derive a system of algebraic equations for $f_j(\bar{x} = 0)$ as follows (for details see Appendix)

$$[\mathbf{M}]\{\mathbf{F}(\bar{x} = 0)\} = \{\mathbf{B}\}. \quad (32)$$

The characteristic quantities describing the heat transfer at the channel walls are

$$q_w = -\lambda(\text{grad } T)_w = h(T_w - T_m), \quad (33)$$

$$T_m = [1/(v_{x,m}A)]\int_A T v_x\,dA, \quad (34)$$

$$Nu = hd_h/\lambda = 2R(\partial T/\partial r)_w/(T_w - T_m) = 2(\partial\bar{T}/\partial\bar{r})_w/(\bar{T}_w - \bar{T}_m), \quad (35)$$

$$(\partial\bar{T}/\partial\bar{r})_{\bar{r}=1} = -\sum_j 2j(j+1)f_jC_j, \quad (36)$$

$$\bar{T}_w = \sum_j f_j[1 - C_j(2j+1)], \quad (37)$$

$$\bar{T}_m = 2(1+2/m)\sum_j f_j\left[\frac{m}{2(m+2)} - C_j\left\{\frac{m(j+1)^2}{(2j+2)(m+2j+2)} - \frac{mj^2}{(2j+4)(m+2j+4)}\right\}\right]. \quad (38)$$

3. RESULTS

To investigate the influence of Biot number on the heat transfer in the thermal entrance region, equations (31) and (32) were solved by employing the standard Crank–Nicolson procedure for the following velocity profiles:

Case 1: uniform velocity profile, slug flow, $m \to \infty$,
Case 2: fully developed viscous velocity profile, $m = 2$,
Case 3: developing velocity profile from case 1 to case 2.

To assess the accuracy of the results obtained in this paper, the special case of $Bi \to \infty$, was analyzed for different number of terms in temperature approximation (11). This special case corresponds to the situation of the constant wall temperature and is well studied by different authors employing different methods. The local Nusselt numbers according to equation (35) for $Bi \to \infty$ were compared with the exact series solutions given by Tao [13] for the slug flow and given by Sellars *et al.* [14] for the fully developed viscous flow. It was found that the accuracy of the results depends strongly upon the number of terms considered in equation (11): closer a location to the channel entrance, larger the numer of terms needed to describe a good approximation for this location. As a weak point of the present method of solution, it was noted that with increasing number of terms in equation (11), the determinant of matrix $[M]$ of equation (32) tends to zero. This means a difficulty in solving equation (32). Consequently, the first eight terms are considered in the approximate solution (11). From Table 1, one can learn that the present local Nusselt numbers based on the

Table 1. Local Nusselt numbers for $Bi \to \infty$ at various values of Pr

$\bar{x}/Pe$	$\bar{v}_x = 2(1-\bar{r}^2)$ [14]	①	Developing flow $Pr = 10$ $v_r \neq 0$	$Pr = 10$ $v_r = 0$	$Pr = 1$ $v_r \neq 0$	$Pr = 1$ $v_r = 0$	$Pr = 0.7$ $v_r \neq 0$	$Pr = 0.7$ $v_r = 0$	$Pr = 0.1$ $v_r \neq 0$	$Pr = 0.1$ $v_r = 0$	$\bar{v}_x = 1$ ④	$\bar{v}_x = 1$ [13]
0.0001	34.86	41.57	45.19	65.50	54.39	67.46	56.25	67.70	65.30	68.50	69.28	111.5
0.0002	28.17	33.48	36.47	57.53	46.69	61.92	48.71	62.29	58.58	63.66	64.68	81.23
0.0005	20.62	20.60	23.45	38.24	33.04	47.41	34.95	48.32	44.86	51.64	53.14	52.07
0.001	16.25	15.19	16.46	22.46	22.33	31.66	23.70	32.91	31.81	38.06	40.36	37.32
0.002	12.81	12.66	13.27	14.99	15.75	19.45	16.38	20.30	20.93	24.87	27.72	26.87
0.005	9.395	9.503	9.531	10.67	11.21	12.88	11.56	13.26	13.78	15.45	17.61	17.67
0.01	7.470	7.456	7.326	7.971	8.359	9.528	8.609	9.814	10.25	11.39	13.09	13.07
0.02	6.002	5.997	5.885	6.168	6.387	7.075	6.551	7.278	7.709	8.448	9.881	9.883
0.05	4.641	4.640	4.644	4.677	4.724	5.042	4.814	5.156	5.563	5.947	7.233	7.237
0.1	4.005	4.002	4.003	4.013	4.010	4.149	4.061	4.222	4.624	4.831	6.177	6.179
0.2	3.710	3.709	3.709	3.710	3.712	3.736	3.723	3.768	4.154	4.242	5.815	5.817
0.5	3.657	3.657	3.657	3.658	3.657	3.658	3.658	3.658	3.879	3.899	5.783	5.783
1.0	3.657	3.657	3.657	3.657	3.657	3.657	3.657	3.657	3.743	3.745	5.783	5.783

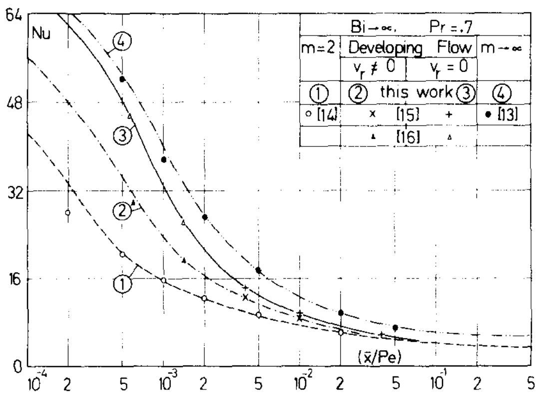

FIG. 1. Comparison of local Nusselt numbers for $Bi \to \infty$ and $Pr = 0.7$.

first eight terms in equation (11) agree well with the results based on the first fifty eigenvalues given in [13] and [14] for $(\bar{x}/Pe) \geq 0.0005$. This accuracy seems to be reasonable for the practical engineering purposes.

In Fig. 1, the present local Nusselt numbers are compared with the results obtained earlier for the developed and developing velocity fields, $Bi \to \infty$ and $Pr = 0.7$. For this purpose, the results of Ulrichson and Schmitz [15] and Hwang and Sheu [16] are employed. They used the finite difference technique to solve the energy equation. They refined the analysis of Kays [17] by utilizing the axial velocity component of Langhaar solution and subsequently the radial component from the continuity equation to estimate the influence of the radial convection and by introducing finer mesh sizes. It seems that the abscissa of the diagrams showing local Nusselt numbers in [15] is misprinted; it is supposed to be $(\bar{x}/Pe)/4$ instead of $4(\bar{x}/Pe)$. From Fig. 1, one can conclude a satisfactory agreement of the present results with the results reported in [15] and [16].

Figure 1 and Table 1 illustrate the influence of the radial convection on the heat-transfer coefficient distinctly. They indicate that the local Nusselt number in the thermal entrance region, specially near the channel entrance, is overestimated significantly, if the radial convection term in the energy equation is neglected. As $(\bar{x}/Pe)$ decreases, the difference between the local Nusselt number considering the radial convection and that neglecting it becomes pronounced, for instance

$$\frac{Nu[(\bar{x}/Pe) = 0.001,\ Bi \to \infty,\ Pr = 0.7,\ \text{without radial convection}]}{Nu[(\bar{x}/Pe) = 0.001,\ Bi \to \infty,\ Pr = 0.7,\ \text{with radial convection}]} = 32.91/23.70 = 1.39.$$

This influence of the radial convection can be explained as follows: near the channel entrance, the fluid mean temperature is nearly the same in both the cases but the wall heat flux given by $(\partial \bar{T}/\partial \bar{r})_w$ in equation (35) is considerably larger than the actual rate of heating, because the continuity equation is not satisfied locally, if the radial convection is neglected.

If the velocity and temperature fields are developing simultaneously, the axial velocity profile remains always in the range limited by $\bar{v}_{x,0} = 1$ and $\bar{v}_{x,d} = 2(1-\bar{r}^2)$. Consequently, for any given value of $(\bar{x}/Pe)$, one expects Nu (slug flow) $\geq Nu \geq Nu$ (fully developed flow).

In the case of $Pr \to 0$, which is approximately valid for liquid metals, one can assume uniform velocity profile. In the case of $Pr \to \infty$, which is nearly valid for very viscous fluids (oils), one can assume fully developed velocity profile. This influence of Prandtl number on the local Nusselt number can be inferred from Table 1 for $Bi \to \infty$.

The other extreme case of $Bi \to 0$, which corresponds to the situation of the constant wall heat flux, cannot be treated with the approximation (11), as $C_j \to 0$. For this particular case, which is well investigated by different authors employing different methods, one has to formulate another approximation, for instance as in [12].

Tables 2–5 show the local Nusselt numbers for the different values of Pr and Bi and also illustrate the influence of the radial convection. The comparison of the present local Nusselt numbers with the results given by Hsu [4], who solved eigenvalue problem numerically assuming a fully developed viscous flow

Table 2. Local Nusselt numbers for $Bi = 100$ at several values of Pr

$\bar{x}/Pe$	$m = 2$	Developing flow $Pr = 10$ $v_r \neq 0$	$Pr = 10$ $v_r = 0$	$Pr = 1$ $v_r \neq 0$	$Pr = 1$ $v_r = 0$	$Pr = 0.7$ $v_r \neq 0$	$Pr = 0.7$ $v_r = 0$	$Pr = 0.1$ $v_r \neq 0$	$Pr = 0.1$ $v_r = 0$	$m \to \infty$
0.0001	43.01	49.74	69.97	58.84	72.63	60.67	72.76	70.48	73.76	74.39
0.0002	33.85	40.53	60.49	50.70	65.99	52.64	66.38	63.08	68.19	69.02
0.0005	21.37	26.49	39.31	36.27	49.54	38.20	50.65	48.34	54.77	56.65
0.001	15.87	18.27	23.38	24.78	32.99	26.24	34.40	34.64	40.27	43.38
0.002	12.98	14.04	15.57	17.05	20.42	17.81	21.40	22.93	26.45	30.17
0.005	9.621	9.837	10.83	11.64	13.21	12.04	13.65	14.54	16.13	18.78
0.01	7.558	7.501	8.081	8.597	9.690	8.874	10.00	10.64	11.72	13.71
0.02	6.061	5.977	6.237	6.514	7.172	6.691	7.387	7.914	8.627	10.23
0.05	4.672	4.677	4.719	4.776	5.084	4.872	5.204	5.649	6.025	7.385
0.1	4.026	4.027	4.037	4.037	4.172	4.090	4.248	4.666	4.870	6.259
0.2	3.722	3.723	3.726	3.723	3.751	3.740	3.784	4.176	4.263	5.869
0.5	3.669	3.669	3.669	3.669	3.669	3.669	3.669	3.892	3.914	5.832
1.0	3.666	3.666	3.666	3.667	3.667	3.667	3.667	3.750	3.763	5.832

Table 3. Local Nusselt numbers for $Bi = 10$ at several values of Pr

$\bar{x}/Pe$	$m = 2$	Developing flow $Pr = 10$ $v_r \neq 0$	$Pr = 10$ $v_r = 0$	$Pr = 1$ $v_r \neq 0$	$Pr = 1$ $v_r = 0$	$Pr = 0.7$ $v_r \neq 0$	$Pr = 0.7$ $v_r = 0$	$Pr = 0.1$ $v_r \neq 0$	$Pr = 0.1$ $v_r = 0$	$m \to \infty$
0.0001	44.61	56.26	77.03	65.25	80.63	67.68	81.73	82.09	83.71	85.03
0.0002	35.60	46.28	64.20	56.10	71.50	58.64	72.91	72.49	76.17	78.49
0.0005	24.02	31.79	40.98	41.27	52.38	43.34	53.88	55.19	59.74	64.09
0.001	18.12	22.52	26.10	29.69	35.74	31.26	37.23	40.59	44.19	50.35
0.002	14.31	16.41	17.73	20.75	23.39	21.72	24.49	28.14	30.31	37.00
0.005	10.40	11.02	11.80	13.43	14.71	13.97	15.30	17.49	18.60	23.70
0.01	8.154	8.276	8.750	9.738	10.62	10.10	11.01	12.47	13.27	17.00
0.02	6.469	6.471	6.684	7.217	7.777	7.441	8.033	9.043	9.594	12.36
0.05	4.924	4.927	4.972	5.126	5.396	5.247	5.543	6.204	6.517	8.518
0.1	4.193	4.194	4.207	4.244	4.365	4.307	4.450	4.974	5.160	6.942
0.2	3.838	3.838	3.841	3.853	3.873	3.866	3.908	4.348	4.428	6.310
0.5	3.763	3.763	3.763	3.764	3.764	3.764	3.764	4.008	4.025	6.224
1.0	3.763	3.763	3.763	3.763	3.763	3.763	3.763	3.855	3.857	6.224

Table 4. Local Nusselt numbers for $Bi = 5$ at several values of Pr

$\bar{x}/Pe$	$m = 2$	Developing flow $Pr = 10$ $v_r \neq 0$	$Pr = 10$ $v_r = 0$	$Pr = 1$ $v_r \neq 0$	$Pr = 1$ $v_r = 0$	$Pr = 0.7$ $v_r \neq 0$	$Pr = 0.7$ $v_r = 0$	$Pr = 0.1$ $v_r \neq 0$	$Pr = 0.1$ $v_r = 0$	$m \to \infty$
0.0005	24.50	31.93	40.19	42.16	52.58	44.31	55.30	58.89	62.98	67.72
0.001	18.68	22.99	26.28	30.56	36.18	32.40	38.21	43.45	46.66	53.19
0.002	14.75	16.81	18.12	21.58	24.02	22.89	25.53	30.11	32.19	38.89
0.005	10.77	11.39	12.16	14.09	15.30	14.76	16.00	18.72	19.79	25.32
0.01	8.468	8.590	9.062	10.26	11.07	10.70	11.53	13.34	14.07	18.35
0.02	6.705	6.715	6.949	7.608	8.128	7.872	8.405	9.651	10.15	13.41
0.05	5.110	5.113	5.149	5.368	5.621	5.504	5.778	6.576	6.858	9.255
0.1	4.329	4.331	4.344	4.407	4.524	4.479	4.611	5.223	5.382	7.473
0.2	3.935	3.937	3.944	3.965	3.978	3.974	4.017	4.495	4.572	6.684
0.5	3.845	3.846	3.847	3.846	3.847	3.846	3.847	4.105	4.122	6.547
1.0	3.844	3.844	3.845	3.844	3.844	3.844	3.845	3.941	3.950	6.547

Table 5. Local Nusselt numbers for $Bi = 2$ at several values of Pr

$\bar{x}/Pe$	$m = 2$	Developing flow $Pr = 10$ $v_r \neq 0$	$Pr = 10$ $v_r = 0$	$Pr = 1$ $v_r \neq 0$	$Pr = 1$ $v_r = 0$	$Pr = 0.7$ $v_r \neq 0$	$Pr = 0.7$ $v_r = 0$	$Pr = 0.1$ $v_r \neq 0$	$Pr = 0.1$ $v_r = 0$	$m \to \infty$
0.001	20.55	26.09	30.65	34.99	41.72	34.96	42.33	49.72	50.38	54.41
0.002	15.82	18.69	20.06	23.87	26.65	24.62	27.60	33.26	33.71	39.32
0.005	11.43	12.36	13.08	15.22	16.45	15.74	16.92	20.13	20.65	25.68
0.01	8.938	9.287	9.691	11.02	11.81	11.43	12.17	14.32	14.71	19.02
0.02	7.112	7.201	7.394	8.159	8.630	8.429	8.894	10.39	10.71	14.26
0.05	5.389	5.391	5.452	5.760	5.981	5.884	6.119	7.118	7.291	10.09
0.1	4.574	4.578	4.579	4.684	4.774	4.756	4.873	5.621	5.721	8.181
0.2	4.120	4.129	4.133	4.165	4.174	4.171	4.210	4.759	4.823	7.289
0.5	4.001	4.003	4.004	4.003	4.005	4.003	4.005	4.285	4.302	7.088
1.0	4.000	4.000	4.000	4.000	4.000	4.000	4.000	4.105	4.108	7.087

and calculated the first twelve eigenvalues and presented the local Nusselt numbers for $(\bar{x}/Pe) \geqslant 0.02$ and $Bi = 2$, 10 and 100, indicated a good agreement. The results for the developing flow cannot be compared, since to the author's knowledge no similar analysis is reported in the literature. In case of finite Biot number, qualitatively the same influence of the radial convection and of Prandtl number on the local Nusselt number can be observed as in the case of $Bi \to \infty$.

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APPENDIX

The elements of the matrices and vectors occurring in equations (31) and (32) are listed below.

N: Number of terms considered in approximation (11).

$$i = 1, 2, \dots N.$$

$$j = 1, 2, \dots N.$$

$$i_1 = i+1,\ j_1 = j+1.$$

$$B(j) = 1/2 - C_j(1+2/m_0)\left[\frac{(j+1)^2 m_0}{(2j+2)(m_0+2j+2)} - \frac{j^2 m_0}{(2j+4)(m_0+2j+4)}\right].$$

$$\begin{aligned} M(i,j) = 1/2 + (1+2/m_0)\Bigg[& -C_i\left\{\frac{i_1^2 m_0}{(2i+2)(m_0+2i+2)} - \frac{i^2 m_0}{(2i+4)(m_0+2i+4)}\right\} \\ & - C_j\left\{\frac{j_1^2 m_0}{(2j+2)(m_0+2j+2)} - \frac{j^2 m_0}{(2j+4)(m_0+2j+4)}\right\} \\ & + C_i C_j\left\{\frac{(i_1 j_1)^2 m_0}{(2i+2j+2)(m_0+2i+2j+2)} - \frac{(i^2 j_1^2 + j^2 i_1^2) m_0}{(2i+2j+4)(m_0+2i+2j+4)} \right. \\ & \left. + \frac{(ij)^2 m_0}{(2i+2j+6)(m_0+2i+2j+6)}\right\}\Bigg]. \end{aligned}$$

$$W(i,j) = -4C_j(j_1 j)^2\left[\frac{2}{2j(2j+2)} - \frac{2C_i}{(2i+2j+2)} \times \left\{\frac{i_1^2}{(2i+2j)} - \frac{i^2}{(2i+2j+4)}\right\}\right].$$

$D(i,j) = M(i,j)$, if m_0 is replaced by m.

$$Q_1 = 1/(2j+2) - 1/(m+2j+2) - m/(m+2j+2)^2,$$

$$Q_2 = 1/(2j+4) - 1/(m+2j+4) - m/(m+2j+4)^2,$$

$$Q_3 = 1/(2i+2j+2) - 1/(m+2i+2j+2) - m/(m+2i+2j+2)^2,$$

$$Q_4 = 1/(2i+2j+4) - 1/(m+2i+2j+4) - m/(m+2i+2j+4)^2,$$

$$Q_5 = 1/(2i+2j+6) - 1/(m+2i+2j+6) - m/(m+2i+2j+6)^2,$$

$$\begin{aligned} V(i,j) = C_j\{ & 2(j+1)^2 j Q_1 - j^2(2j+2)Q_2 \\ & -(j+1)^2 2j C_i[(i+1)^2 Q_3 - i^2 Q_4] \\ & + j^2(2j+2)C_i[(i+1)^2 Q_4 - i^2 Q_5]\}. \end{aligned}$$

ETABLISSEMENT SIMULTANE DES CHAMPS LAMINAIRES DE VITESSE ET DE TEMPERATURE DANS UN TUBE CIRCULAIRE AVEC CONDITIONS AUX LIMITES DE TROISIEME ESPECE SUR LA TEMPERATURE

Résumé—Afin d'étudier l'influence de l'établissement du régime d'écoulement dans un tube circulaire sur la convection thermique forcée laminaire, on résout l'équation non-linéaire de quantité de mouvement et l'équation linéaire de la chaleur par la méthode, de Galerkin–Kantorowich, du calcul des variations. Moyennant des hypothèses de propriétés physiques du fluide constantes, d'une diffusion axiale négligeable et de conditions aux limites de troisième espèce sur la température, on a obtenu une solution explicite de vitesse et une solution semi-analytique pour la température. Il ressort que les nombres de Nusselt locaux obtenus en considérant la convection radiale ou en la négligeant peuvent présenter d'importantes différences qui dépendent des nombres de Biot et de Prandtl.

GLEICHZEITIGE AUSBILDUNG DER LAMINAREN GESCHWINDIGKEITS- UND TEMPERATURFELDER IN EINEM KREISFÖRMIGEN KANAL FÜR DIE RANDBEDINGUNG DRITTER ART

Zusammenfassung—Mit Hilfe der Galerkin–Kantorowich Methode wurde die nichtlineare Bewegungs- und die lineare Energiegleichung sukzessiv gelöst, um den Einfluß der sich ausbildenden Strömung in einem kreisförmigen Rohr bei Wärmeübertragung unter laminarer erzwungener Konvektion zu analysieren. Unter Annahme konstanter Stoffwerte vernachlässigbarer axialer Diffusion und der Randbedingung dritter Art wurde eine Lösung in geschlossener Form für die Geschwindigkeit und eine halbanalytische Lösung für die Temperatur hergeleitet. Es zeigte sich, daß zwischen der örtlichen Nusselt–Zahl bei Berücksichtigung der radialen Konvektion und jener bei Vernachlässigung dieser Konvektion, abhängig von der Bio– und Prandtl–Zahl beträchtliche Unterschiede auftreten können.

ОДНОВРЕМЕННОЕ ФОРМИРОВАНИЕ ПОЛЕЙ ЛАМИНАРНОЙ СКОРОСТИ И ТЕМПЕРАТУРЫ В ТРУБЕ КРУГЛОГО СЕЧЕНИЯ ПРИ ТЕМПЕРАТУРНОМ ГРАНИЧНОМ УСЛОВИИ ТРЕТЬЕГО РОДА

Аннотация — Используя метод вариационного исчисления Галеркина-Канторовича, последовательно решены нелинейное уравнение количества движения и линейное уравнение энергии, с помощью которых анализируется влияние развивающегося течения в трубе круглого сечения на теплообмен при ламинарной вынужденной конвекции. В предположении постоянных свойств жидкости, незначительной аксиальной диффузии и температурного граничного условия третьего рода, получено решение в замкнутом виде для скорости и температуры. Из проведенного анализа следует, что в зависимости от чисел Био и Прандтля локальные числа Нуссельта при учете радиальной конвекции и без нее могут значительно отличаться.